Integral Solutions - Integral Four
Barrier Integral Solutions

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We will define the variable μ to be expected return mean, the variable ϕ to be the dividend yield, and the variable σ to be expected return volatility. We will define the variables α and υ to be the mean and variance, respectively, of a Brownian motion over the time interval $[0,t]$. The equations for the Brownian motion's mean and variance are...

$$\alpha = \left(\mu - \phi - \frac{1}{2}\sigma^2\right)t \text{ ...and... } \upsilon = \sigma^2 t \tag{1}$$

We will define the function $a(m,w)$ to be the joint distribution function of the value of a Brownian motion (w) at time t and its maximum or minimum (m) over the time interval $[0,t]$. The equation for the joint distribution function is... [1]

$$a(m,w) = \frac{2\,(w-2\,m)}{\upsilon\sqrt{2\,\pi\,\upsilon}}\,\text{Exp}\left\{\frac{\alpha\,w}{\upsilon} - \frac{\alpha^2}{2\,\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\} \tag{2}$$

The equation for the anti-derivative of Equation (2) above is...

$$\frac{\delta}{\delta m}\sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{\frac{\alpha\,w}{\upsilon} - \frac{\alpha^2}{2\,\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\} = a(m,w) \tag{3}$$

The equation for the anti-derivative of the product of Equation (2) above and the exponential of the random variable w is...

$$\frac{\delta}{\delta m}\sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{w + \frac{\alpha\,w}{\upsilon} - \frac{\alpha^2}{2\,\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\} = \text{Exp}\left\{w\right\}a(m,w) \tag{4}$$

We will define the function $CNDF(x, mean, variance)$ to be the cumulative normal distribution function of the normally-distributedrandom variable x. The equation for the cumulative normal distribution function is...

$$CNDF(x,\mu,\upsilon) = \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w-\mu)^2\right\}\delta w \text{ ...where... } x \sim N\Big[\mu,\upsilon\Big] \tag{5}$$

The Excel function for the cumulative normal distribution function as defined by Equation (5) above is...

$$CNDF(x,\mu,\upsilon) = NORMDIST(x,\mu,\sqrt{\upsilon},True) \tag{6}$$

Integral Definitions

Using Equation (2) above we want to solve integral one, which is defined as...

$$\begin{aligned}
I_1 &= \int\limits_{w=c}^{w=d}\int\limits_{m=a}^{m=b} a(m,w)\,\delta m\,\delta w \\
&= \int\limits_{w=c}^{w=d}\sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=a}^{m=b}\delta w \\
&= \int\limits_{w=c}^{w=d}\sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,b-w)^2\right\}\delta w - \int\limits_{w=c}^{w=d}\sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\text{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,a-w)^2\right\}\delta w
\end{aligned} \tag{7}$$

Using Equation (2) above we want to solve integral two, which is defined as...

$$I_2 = \int\limits_{w=c}^{w=d} \int\limits_{m=a}^{m=b} \mathrm{Exp}\left\{w\right\} a(m,w)\,\delta w$$
$$= \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\Bigg[_{m=a}^{m=b}\,\delta w$$
$$= \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,b-w)^2\right\}\delta w - \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,a-w)^2\right\}\delta w \tag{8}$$

Integral Solutions

A. Using Equation (7) above we want to solve the following integral...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \;\;\text{...where...}\;\; m = \text{constant} \tag{9}$$

Note that we can rewrite Equation (9) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\left(w^2 + 4\,m^2 - 4\,m\,w - 2\,\alpha\,w + \alpha^2\right)\right\}\delta w \tag{10}$$

We will make the following definition...

$$(w - \alpha - 2\,m)^2 = w^2 + 4\,m^2 - 4\,m\,w - 2\,\alpha\,w + \alpha^2 + 4\,m\,\alpha \tag{11}$$

Using Equation (11) above we can rewrite Equation (10) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w-\alpha-2\,m)^2\right\}\mathrm{Exp}\left\{\frac{1}{2\,\upsilon}\,4\,m\,\alpha\right\}\delta w$$
$$= \mathrm{Exp}\left\{\frac{2\,m\,\alpha}{\upsilon}\right\}\int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w-(\alpha+2\,m))^2\right\}\delta w \tag{12}$$

Using Equation (5) above the solution to Equation (12) above is...

$$I = \mathrm{Exp}\left\{\frac{2\,m\,\alpha}{\upsilon}\right\}\Big[CNDF(d,\alpha+2\,m,\upsilon) - CNDF(c,\alpha+2\,m,\upsilon)\Big] \tag{13}$$

B. Using Equation (7) above we want to solve the following integral...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m-w)^2\right\}\delta w \;\;\text{...where...}\;\; m = \text{random variable } w \tag{14}$$

Note that we can rewrite Equation (14) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\left(w^2 - 2\,\alpha\,w + \alpha^2\right)\right\}\delta w \tag{15}$$

We will make the following definition...

$$(w-\alpha)^2 = w^2 - 2\,\alpha\,w + \alpha^2 \tag{16}$$

Using Equation (16) above we can rewrite Equation (15) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w-\alpha)^2\right\}\delta w \tag{17}$$

Using Equation (5) above the solution to Equation (17) above is...

$$I = CNDF(d,\alpha,\upsilon) - CNDF(c,\alpha,\upsilon) \tag{18}$$

C. Using Equation (8) above we want to solve the following integral...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\}\delta w \;\;\text{...where...}\;\; m = \text{constant} \tag{19}$$

Note that we can rewrite Equation (19) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w^2 + 4\,m^2 - 4\,m\,w - 2\,\alpha\,w - 2\,\upsilon\,w + \alpha^2)\right\}\delta w \tag{20}$$

We will make the following definition...

$$(w - \alpha - \upsilon - 2\,m)^2 = w^2 + 4\,m^2 - 4\,m\,w - 2\,\alpha\,w - 2\,\alpha\,w + \alpha^2 + 4\,m\,\alpha + 4\,m\,\upsilon + 2\,\alpha\,\upsilon + \upsilon^2 \tag{21}$$

Using Equation (21) above we can rewrite Equation (20) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w - \alpha - \upsilon - 2\,m)^2\right\}\mathrm{Exp}\left\{\frac{1}{2\,\upsilon}\,(4\,m\,\alpha + 4\,m\,\upsilon + 2\,\alpha\,\upsilon + \upsilon^2)\right\}\delta w$$

$$= \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2} + 2\,m + \frac{2\,m\,\alpha}{\upsilon}\right\}\int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w - (\alpha + \upsilon + 2\,m))^2\right\}\delta w \tag{22}$$

Using Equation (5) above the solution to Equation (22) above is...

$$I = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2} + 2\,m + \frac{2\,m\,\alpha}{\upsilon}\right\}\Big[CNDF(d,\alpha + \upsilon + 2\,m,\upsilon) - CNDF(c,\alpha + \upsilon + 2\,m,\upsilon)\Big] \tag{23}$$

D. Using Equation (8) above we want to solve the following integral...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\}\delta w \;\;\text{...where...}\;\; m = \text{random variable } w \tag{24}$$

Note that we can rewrite Equation (24) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w^2 - 2\,\alpha\,w - 2\,\upsilon\,w + \alpha^2)\right\}\delta w \tag{25}$$

We will make the following definition...

$$(w - \alpha - \upsilon)^2 = w^2 - 2\,\alpha\,w + \alpha^2 + 2\,\alpha\,\upsilon + \upsilon^2 \tag{26}$$

Using Equation (26) above we can rewrite Equation (25) above as...

$$I = \int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w - \alpha - \upsilon)^2\right\}\mathrm{Exp}\left\{\frac{1}{2\,\upsilon}\,(2\,\alpha\,\upsilon + \upsilon^2)\right\}\delta w$$

$$= \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2}\right\}\int\limits_{w=c}^{w=d} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{-\frac{1}{2\,\upsilon}\,(w - (\alpha + \upsilon))^2\right\}\delta w \tag{27}$$

Using Equation (5) above the solution to Equation (27) above is...

$$I = \mathrm{Exp}\left\{\alpha + \frac{\upsilon}{2}\right\}\Big[CNDF(d,\alpha + \upsilon,\upsilon) - CNDF(c,\alpha + \upsilon,\upsilon)\Big] \tag{28}$$

Examples

For the examples below we will assume that $\alpha = 0.05$ and $\upsilon = 0.18$.

Example 1: We want to solve the following integral...

$$I = \int\limits_{w=0.25}^{w=\infty} \int\limits_{m=-0.20}^{m=0.00} a(m, w)\,\delta m\,\delta w = \int\limits_{w=0.25}^{w=\infty} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\}\Bigg[_{m=-0.20}^{m=0.00} \delta w \tag{29}$$

Using Equation (7) above we can rewrite Equation (29) above as the difference between the following two integrals...

$$I_a = \int\limits_{w=0.25}^{w=\infty} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\} \delta w \text{ ...where... } m = 0$$

$$I_b = \int\limits_{w=0.25}^{w=\infty} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{\frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\} \delta w \text{ ...where... } m = -0.20 \tag{30}$$

Using Equation (13) the solutions to Equation (30) above is...

$$\begin{aligned} I_a &= \mathrm{Exp}\left\{\frac{2 \times 0 \times 0.05}{0.18}\right\}\Bigg[CNDF(\infty, 0.05 + 2 \times 0, 0.18) - CNDF(0.25, 0.05 + 2 \times 0, 0.18)\Bigg] \\ &= 1 - CNDF(0.25, 0.05 + 2 \times 0, 0.18)\Bigg] \\ &= 0.31868 \end{aligned} \tag{31}$$

$$\begin{aligned} I_a &= \mathrm{Exp}\left\{\frac{2 \times -0.20 \times 0.05}{0.18}\right\}\Bigg[CNDF(\infty, 0.05 + 2 \times -0.20, 0.18) - CNDF(0.25, 0.05 + 2 \times -0.20, 0.18)\Bigg] \\ &= 1 - CNDF(0.25, 0.05 + 2 \times -0.20, 0.18)\Bigg] \\ &= 0.07038 \end{aligned} \tag{32}$$

Using Equations (31) and (32) the answer to the problem is...

$$I = 0.31868 - 0.07038 = 0.24830 \tag{33}$$

Example 2: We want to solve the following integral...

$$I = \int\limits_{w=-0.40}^{w=0.00} \int\limits_{m=-0.80}^{m=w} \mathrm{Exp}\Big\{w\Big\}\, a(m, w)\,\delta m\,\delta w = \int\limits_{w=-0.40}^{w=0.00} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\}\Bigg[_{m=-0.80}^{m=w} \delta w \tag{34}$$

Using Equation (8) above we can rewrite Equation (34) above as the difference between the following two integrals...

$$I_a = \int\limits_{w=-0.40}^{w=0.00} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\} \delta w \text{ ...where... } m = w$$

$$I_b = \int\limits_{w=-0.40}^{w=0.00} \sqrt{\frac{1}{2\,\pi\,\upsilon}}\,\mathrm{Exp}\left\{w + \frac{\alpha w}{\upsilon} - \frac{\alpha^2}{2\upsilon} - \frac{1}{2\,\upsilon}(2\,m - w)^2\right\} \delta w \text{ ...where... } m = -0.80 \tag{35}$$

Equation (28)

$$I_a = \mathrm{Exp}\left\{0.05 + \frac{0.18}{2}\right\}\Bigg[CNDF(0, 0.05 + 0.18, 0.18) - CNDF(-0.40, 0.05 + 0.18, 0.18)\Bigg] \tag{36}$$

Equation (23)

$$I_b = \text{Exp}\left\{0.05 + \frac{0.18}{2} + 2 \times 0 - 0.80 + \frac{2 \times -0.80 \times 0.05}{0.18}\right\}$$
$$\left[CNDF(0, \alpha + 0.18 + 2 \times -0.80, 0.18) - CNDF(-0.40, 0.05 + 0.18 + 2 \times -0.80, 0.18)\right] \quad (37)$$

References

[1] Integral Solutions - Integral Two (Joint Distribution Anti-Derivative), Schurman